Problem Set 3
Optical Waveguides and Fibers (OWF)

Exercise 1: Transfer matrix formalism

Consider a stack of alternating layers consisting of five layers of silicon dioxide ($n_{\text{SiO}_2} = 1.44$ at $\lambda \approx 1.55\,\mu\text{m}$) and four layers of silicon ($n_{\text{Si}} = 3.48$ at $\lambda \approx 1.55\,\mu\text{m}$), as shown in Fig. 1. Let the thicknesses of the layers be $d_{\text{SiO}_2} = 796\,\text{nm}$ and $d_{\text{Si}} = 330\,\text{nm}$. The layer stack is embedded in air ($n_{\text{air}} = 1$) that corresponds to regions 1 and 11 in Fig. 1. In this problem set, we are going to calculate the wavelength-dependent power transmission $\tau(\lambda)$ and power reflection $\rho(\lambda)$, when a plane wave impinges orthogonally on one side of the layer stack.

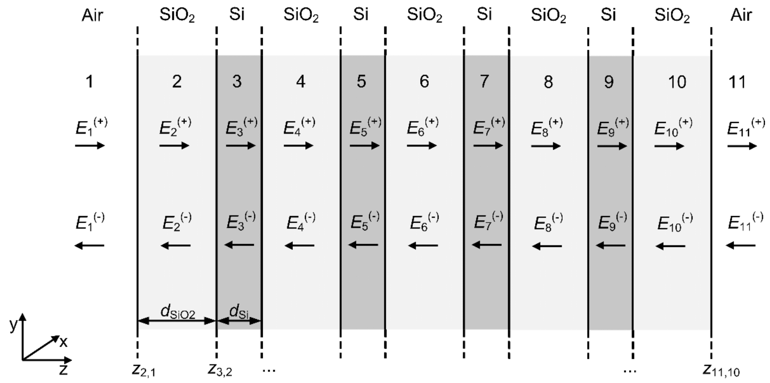

Figure 1: The layer stack and the corresponding coordinate system.

The amplitudes $\hat{E}_\nu^{(\pm)}$ ($\nu = 1 \dots N$) denote the foward-propagating (+) and backward-propagating (-) amplitudes in the ν-th region. The electric field is then defined by:

$$\begin{aligned} \underline{\mathbf{E}}_\nu(z,t) &\equiv \mathbf{e}_x \left[\hat{E}_\nu^{(+)} e^{j(\omega t - n_\nu k_0 z)} + \hat{E}_\nu^{(-)} e^{j(\omega t + n_\nu k_0 z)} \right] \\ &\equiv \mathbf{e}_x \left[E_\nu^{(+)}(z) e^{j\omega t} + E_\nu^{(-)}(z) e^{j\omega t} \right] \end{aligned} \tag{1}$$

where k_0 is the propagation constant in vacuum and $\mathbf{e}_x$ is the unit vector along x representing an x-polarized wave without loss of generality.

a) Consider first a *single* planar boundary located at $z_{2,1}$ separating two semi-infinite regions having indices of refraction n_1 and n_2. Assume that two waves having amplitudes $E_1^{(+)}(z)$ and $E_2^{(-)}(z)$ are impinging on the boundary from the left and the right. Show that the amplitudes of the outgoing waves can be expressed in terms of the ingoing amplitudes by:

$$\begin{pmatrix} E_1^{(-)}(z_{2,1}) \\ E_2^{(+)}(z_{2,1}) \end{pmatrix} = \begin{pmatrix} \frac{n_1-n_2}{n_1+n_2} & \frac{2n_2}{n_1+n_2} \\ \frac{2n_1}{n_1+n_2} & \frac{n_2-n_1}{n_1+n_2} \end{pmatrix} \begin{pmatrix} E_1^{(+)}(z_{2,1}) \\ E_2^{(-)}(z_{2,1}) \end{pmatrix}. \tag{2}$$

To this end, you can make use of the linearity of Maxwell's equations and the equations you derived in the lecture when a *single* plane wave is impinging on a boundary x.

Solution: By using the equations for amplitude reflection and transmission factors for TE-polarized waves, as derived in the lecture notes, and only a single wave impinging from the left, i.e., $E_2^{(-)}(z) = 0$, the outgoing waves can be expressed by:

$$E_1^{(-)}(z_{2,1}) = R_{12} E_1^{(+)}(z_{2,1}) \text{ and } E_2^{(+)}(z_{2,1}) = T_{12} E_1^{(+)}(z_{2,1}),$$

where

$$R_{12} = \frac{n_1 - n_2}{n_1 + n_2} \text{ and } T_{12} = \frac{2n_1}{n_1 + n_2}.$$

Applying the same consideration to the case where only a single wave impinges from the right, i.e., $E_1^{(+)}(z) = 0$, the outgoing waves can be expressed by:

$$E_2^{(+)}(z_{2,1}) = R_{21} E_2^{(-)}(z_{2,1}) \text{ and } E_1^{(-)}(z_{2,1}) = T_{21} E_2^{(-)}(z_{2,1}),$$

where

$$R_{21} = \frac{n_2 - n_1}{n_1 + n_2} \text{ and } T_{21} = \frac{2n_2}{n_1 + n_2}.$$

Finally we use the superposition principle, i.e. the sum of two solutions to Maxwell's equations is also a valid solution. Combining the relations above leads to required scattering matrix shown in Eq. (2).

b) In order to easily describe how the amplitudes propagate through the layer, it is useful to express the amplitudes at the right boundary of the layer in terms of those at the left boundary. Show that the following relations hold:

$$\begin{pmatrix} E_2^{(+)}(z_{2,1}) \\ E_2^{(-)}(z_{2,1}) \end{pmatrix} = \mathbf{T}_{2,1} \begin{pmatrix} E_1^{(+)}(z_{2,1}) \\ E_1^{(-)}(z_{2,1}) \end{pmatrix} \quad (3)$$

$$\text{with} \quad \mathbf{T}_{2,1} = \begin{pmatrix} \frac{n_1+n_2}{2n_2} & \frac{n_2-n_1}{2n_2} \\ \frac{n_2-n_1}{2n_2} & \frac{n_1+n_2}{2n_2} \end{pmatrix}. \quad (4)$$

Solution: The intention here is to express the waves on the right hand side of the interface in terms of the waves on its left hand side. This can be done by reordering the terms of Eq. (2), which leads to the following transmission matrix:

$$\begin{pmatrix} T_{12} - \frac{R_{12}R_{21}}{T_{21}} & \frac{R_{21}}{T_{21}} \\ -\frac{R_{12}}{T_{21}} & \frac{1}{T_{21}} \end{pmatrix}.$$

Using the definitions of the amplitude reflection and transmission coefficients given above, we get the transmission matrix $\mathbf{T}_{2,1}$.

c) Substituting the definitions $E_\nu^{(\pm)}(z) = \hat{E}_\nu^{(\pm)} e^{\mp j n_\nu k_0 z}$, show that the propagation through two layers and the associated interface can be described by:

$$\begin{pmatrix} \hat{E}_2^{(+)} \\ \hat{E}_2^{(-)} \end{pmatrix} = \begin{pmatrix} e^{jn_2k_0z_{2,1}} & 0 \\ 0 & e^{-jn_2k_0z_{2,1}} \end{pmatrix} \mathbf{T}_{2,1} \begin{pmatrix} e^{-jn_1k_0z_{2,1}} & 0 \\ 0 & e^{jn_1k_0z_{2,1}} \end{pmatrix} \begin{pmatrix} \hat{E}_1^{(+)} \\ \hat{E}_1^{(-)} \end{pmatrix}. \quad (5)$$

Solution: Starting from Eq. (3) and inserting the E-field ansatz gives:

$$\begin{pmatrix} \hat{E}_2^{(+)} e^{-jn_2k_0z_{2,1}} \\ \hat{E}_2^{(-)} e^{jn_2k_0z_{2,1}} \end{pmatrix} = \mathbf{T}_{2,1} \begin{pmatrix} \hat{E}_1^{(+)} e^{-jn_1k_0z_{2,1}} \\ \hat{E}_1^{(-)} e^{jn_1k_0z_{2,1}} \end{pmatrix}.$$

The exponential functions can be taken out of the vectors and placed into two 2x2 matrices that left-multiply the respective vectors:

$$\begin{pmatrix} e^{-jn_2k_0z_{2,1}} & 0 \\ 0 & e^{jn_2k_0z_{2,1}} \end{pmatrix} \begin{pmatrix} \hat{E}_2^{(+)} \\ \hat{E}_2^{(-)} \end{pmatrix} = \mathbf{T}_{2,1} \begin{pmatrix} e^{-jn_1k_0z_{2,1}} & 0 \\ 0 & e^{jn_1k_0z_{2,1}} \end{pmatrix} \begin{pmatrix} \hat{E}_1^{(+)} \\ \hat{E}_1^{(-)} \end{pmatrix}$$

Left-multiplying both sides of the last equation by the inverse of the leftmost matrix gives Eq. (5).

d) Finally, generalizing and combining all the previous results, show that one can write for the entire layer stack:

$$\begin{pmatrix} E_{11}^{(+)}(z_{2,1}) \\ E_{11}^{(-)}(z_{2,1}) \end{pmatrix} = \mathbf{M} \begin{pmatrix} E_1^{(+)}(z_{2,1}) \\ E_1^{(-)}(z_{2,1}) \end{pmatrix} \quad (6)$$

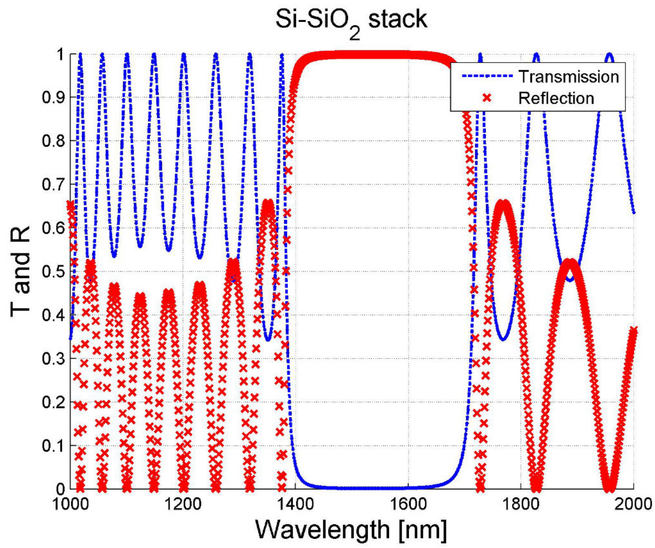

Figure 2: Power transmission and reflection factor as a function of wavelength using the given parameters.

where

$$\mathbf{M} = \mathbf{T}_{N,N-1}\,\mathbf{P}_{N-1}\cdots\,\mathbf{P}_2\,\mathbf{T}_{2,1}$$

is the total transmission matrix, and where

$$\mathbf{P}_\nu = \begin{pmatrix} e^{-jn_\nu k_0 d_\nu} & 0 \\ 0 & e^{jn_\nu k_0 d_\nu} \end{pmatrix}$$

represents the propagator through the ν-th region that has thickness d_ν.

Solution: This can be shown recursively by expressing the waves in the last layer as a function of the waves in the second last layer. In the same way, the waves in the second last layer can be expressed as a function of the waves in the third last layer, and so on. Continuing in this fashion, and inserting every respective expression, will result in the matrix multiplication shown in Eq. (6).

e) The relation between the fields at the beginning and the end of the layer stack is given by

$$\begin{pmatrix} E_\nu^{(+)}(z_{\nu,\nu-1}) \\ E_\nu^{(-)}(z_{\nu,\nu-1}) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_1^{(+)}(z_{2,1}) \\ E_1^{(-)}(z_{2,1}) \end{pmatrix}.$$

With the help of a numerical language (e.g., MATLAB), plot the power transmission coefficient from right to left $\tau(\lambda) = \frac{|\hat{E}_1^{(-)}|^2}{|\hat{E}_{11}^{(-)}|^2} = 1/|M_{22}|^2$ and the power reflection coefficient $\rho(\lambda) = \frac{|E_{11}^{(+)}|^2}{|E_{11}^{(-)}|^2} = |M_{12}|^2/|M_{22}|^2$ in the range $\lambda = 1000\,\text{nm}$ to $2000\,\text{nm}$. Notice that the latter equations for τ and ρ are obtained by setting $\hat{E}_1^{(+)} = 0$.
Hint: For simplicity, consider a constant refractive index over the whole wavelength range.

Solution: The layer stack is a reciprocal structure. That means that power transmission and reflection factors do not depend on the side from which a wave is impinging (from the right or from the left.) In this case we consider that light impinges from the right, i.e., $\hat{E}_1^{(+)} = 0$. For the given set of parameters the power transmission and reflection are plotted as functions of wavelength using MATLAB, and shown in Fig. 2.

f) Finally, set the value of the refractive index of the silicon dioxide in your numerical code to $n_{\mathrm{SiO_2}} = 1$ (this is equivalent to substituting the silicon dioxide with air), and take a large number of layers (for example 30.) Assume new layer thicknesses of $d_{\mathrm{SiO_2}} = 500\,\text{nm}$ and $d_{\mathrm{Si}} = 1\,\text{nm}$, and repeat the transmission and reflection plots in the range $\lambda = 200\,\text{nm}$ to $3000\,\text{nm}$. What are the largest wavelengths at which you observe a reduced transmission? Can you provide an explanation of this phenomenon?

Solution: A peak reflection occurs for every wavelength where an integer multiple of half the wavelength is equal to the pitch of the dielectric layer stack: $m\lambda/2 = d$. This can be physically

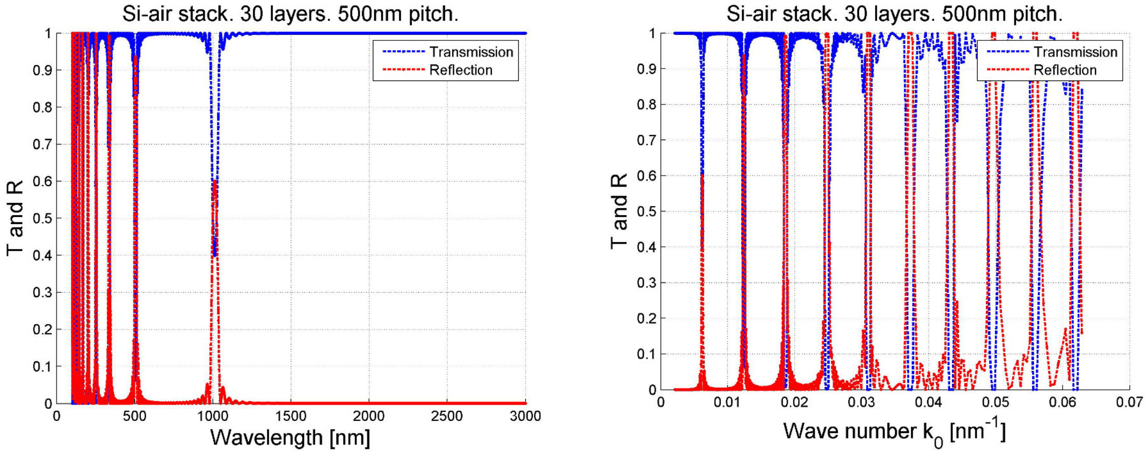

Figure 3: Power transmission and reflection factors as functions of wavelength and wave number using the given parameters.

explained in such a way that the the reflected wave at one layer adds up constructively with the reflected wave of previous and/or subsequent layers. The result is shown in Fig. 3 on the left.

g) What do you observe if you plot the transmission and the reflection coefficients as functions of the wavenumber k_0?

Solution: See Fig. 3 on the right.

Questions and Comments:

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```matlab
% PS_03_OWF_solution
% Part e

clear all;
close all;
LetterSize = 15;
LineSize = 1.5;

T = @(n2, n1) [(n1+n2)/(2*n2) (n2-n1)/(2*n2); (n2-n1)/(2*n2) (n1+n2)/(2*n2)];
P = @(n, k0, d) [exp(-1i*n.*k0*d) 0; 0 exp(1i*n.*k0*d) ];

d_SiO2 = 798;
d_Si = 330;
n_air = 1;
n_SiO2 = 1.44;
n_Si = 3.48;


i = 1;
for lambda = 1000:1:2000
    k0 = 2*pi/lambda;

    M = T(n_air, n_SiO2)*...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*...
        T(n_SiO2, n_air);

    wavelength(i) = lambda;
    transmission(i) = 1/abs( M(2,2) )^2;
    reflection(i) = abs( M(1,2) )^2/abs( M(2,2) )^2;
    i = i+1;
end

% Plots
figure()
hold on
plot(wavelength, transmission, '--.', 'LineWidth', LineSize )
plot(wavelength, reflection, 'x', 'Color', 'r', 'LineWidth', LineSize )
xlabel('Wavelength [nm]','fontsize',LetterSize)
ylabel('T and R','fontsize',LetterSize)
title('Si-SiO_2 stack','fontsize',LetterSize)
legend('Transmission', 'Reflection')
grid on

h = gcf;
set(gcf, 'PaperPositionMode', 'auto') % Use screen size, that you set with 'Position'
print(h,'-djpeg','fig_5_layers.jpg','-r300') % -r100 is 100pixel/inch.
```

```matlab
% PS_03_OWF_solution
% Parts f and g

clear all;
close all;
LetterSize = 15;
LineSize = 1.5;

T = @(n2, n1) [(n2+n1)/(2*n2) (n2-n1)/(2*n2); (n2-n1)/(2*n2) (n2+n1)/(2*n2)];
P = @(n, k0, d) [exp(-1i*n.*k0*d) 0; 0 exp(1i*n.*k0*d) ];

d_SiO2 = 500; % nm
d_Si = 1; % nm
n_air = 1;
n_SiO2 = 1;
n_Si = 3.48;

i = 1;
for lambda = 100:1:3000
    k0 = 2*pi/lambda;

    M = T(n_air, n_SiO2)*...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
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        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*T(n_SiO2, n_Si)*P(n_Si, k0, d_Si)*T(n_Si, n_SiO2)* ...
        P(n_SiO2, k0, d_SiO2)*...
        T(n_SiO2, n_air);

    wavelength(i) = lambda;
    transmission(i) = 1/abs( M(2,2) )^2;
    reflection(i) = abs( M(1,2) )^2/abs( M(2,2) )^2;
    i = i+1;
end
```

```matlab
%% Plots
figure()
hold on
plot(wavelength, transmission, '--.', 'LineWidth', LineSize )
plot(wavelength, reflection, '--.', 'Color', 'r', 'LineWidth', LineSize )
xlabel('Wavelength [nm]','fontsize',LetterSize)
ylabel('T and R','fontsize',LetterSize)
title('Si-air stack. 30 layers. 500nm pitch.','fontsize',LetterSize)
legend('Transmission', 'Reflection')
grid on

h = gcf;
set(gcf, 'PaperPositionMode', 'auto') % Use screen size, that you set with 'Position'
print(h,'-djpeg','fig_30_layers.jpg','-r300') % -r100 is 100pixel/inch.


%% Plots
k_0 = 2*pi./wavelength;
figure()
hold on
plot(k_0, transmission, '--.', 'LineWidth', LineSize )
plot(k_0, reflection, '--.', 'Color', 'r', 'LineWidth', LineSize )
xlabel('Wave number k_0 [nm^-^1]','fontsize',LetterSize)
ylabel('T and R','fontsize',LetterSize)
title('Si-air stack. 30 layers. 500nm pitch.','fontsize',LetterSize)
legend('Transmission', 'Reflection')
grid on

h = gcf;
set(gcf, 'PaperPositionMode', 'auto') % Use screen size, that you set with 'Position'
print(h,'-djpeg','fig_30_layers_k_0.jpg','-r300') % -r100 is 100pixel/inch.
```